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A comparative study of two-phase flow models relevant to bubble column dynamics

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Multiphase flow modelling is still a major challenge in fluid dynamics and, although many different models have been derived, there is no clear evidence of their relevance to certain flow situations. That is particularly valid for bubbly flows, because most of the studies have considered the case of fluidized beds. In the present study we give a general formulation to five existing models and study their relevance to bubbly flows. The results of the linear analysis of those models clearly show that only two of them are applicable to that case. They both show a very similar qualitative linear stability behaviour. In the subsequent asymptotic analysis we derive an equation hierarchy which describes the weakly nonlinear stability of the models. Their qualitative behaviour up to first order with respect to the small parameter is again identical. A permanent-wave solution of the first two equations of the hierarchy is found. It is shown, however, that the permanent-wave (soliton) solution is very unlikely to occur for the most common case of gas bubbles in water. The reason is that the weakly nonlinear equations are unstable due to the low magnitude of the bulk modulus of elasticity. Physically relevant stabilization can eventually be achieved using some available experimental data. Finally, a necessary condition for existence of a fully nonlinear soliton is derived.

1. Introduction

Multiphase flows occur quite frequently in chemical and mineral processing equipment. The hydrodynamics is often the most neglected part in the design of such equipment. Its scale-up from bench scale to pilot scale and finally to industrial scale is often fraught with problems as the flow features change quite dramatically with changing scale of the equipment. Hence there is a need for developing models that describe the multiphase flows in a scale-invariant fashion; this scaling property should at least remain valid over the range of scales of interest. Volume, time and ensemble averaging procedures have been used to develop the model equations. Since averaging results in loss of information that must be obtained from closure models, there is a need for validation of such multiphase models. This entails obtaining solutions to the model equations using specific forms of closure models and comparing the predictions with experimental data on each of the three scales. The model equations are strongly nonlinear, posing challenges to the numerical algorithms. Experiments are also difficult to perform, particularly when detailed data on spatial distribution of velocity and pressure fields are needed on the same scale as the averaging procedure used to develop the equations. Recent developments in experimental techniques using magnetic resonance imaging (MRI) provide hope that such detailed experimental data might be forthcoming in the future. At present, model discrimination and validation is done by examining certain properties of the model equations such as the onset of instability, and by comparing qualitative features of the flow (such as the bubble formation in a fluidized bed) from full numerical solutions of the multiphase flow model equations. A good example of such an approach is that of Anderson, Sundaresan & Jackson (1995) on fluidized beds, and Jones & Prosperetti (1985) and Prosperetti & Jones (1987) on general forms of two-phase flow model equations.

The primary goal of our effort is to investigate the variations among models that purport to capture the non-stationary behaviour of flow in a bubble column. Bubble columns form an industrially important class of process equipment, very much like the fluidization equipment. The physics of flow in these two processes (the bubble column and the fluidized bed) also share many common features. Our interest in this problem arose out of a need to examine the dynamics of a bubble column using a commercial computational fluid dynamics code. We soon realized the inadequacies of the existing models used in such codes. A close look at the equations solved by many commercial codes, when a multi-fluid model is used, shows that the terms accounting for the added (virtual) mass force and the elastic reaction of the dispersed phase are neglected. Moreover, the pressure is assumed to be equal in both phases. This approach ignores some of the characteristic differences between the single-fluid and multi-fluid models. As mentioned by Biesheuvel & Wijngaarden van (1984) and Stuhmiller (1977), if these approximations are combined with the neglect of the viscous dissipation term, the equations possess imaginary characteristics provided that the velocities of both phases are unequal. The implication of this is that the system is linearly unstable (as shown in § 3) for any value of the void fraction, a fact that contradicts the experimental evidence for fluidized beds and bubbly flows (see Batchelor 1988). Numerically that means that each constituent numerical algorithm will develop instability inherited from the equations. Each disturbance in the flow will grow exponentially until reaching large enough amplitudes for the nonlinear terms to equilibrate it. Thus, as mentioned by Stuhmiller (1977), the solution to such a nonlinear problem can still be bounded although unsteady. On the other hand there is experimental evidence (see Biesheuvel & Gorissen 1990) that a threshold in the volume fraction exists below which the flow is stable. Garg & Pritchett (1975) recognized (in the case of gas-fluidized beds) that the degree of instability is related to the magnitude of the effective bulk modulus of elasticity of the particles.

When bubbly flows are considered it is also important to incorporate into the model the so-called added-mass force which accounts for the loss of momentum of an accelerating particle needed to accelerate the surrounding fluid in the opposite direction. As mentioned by Stuhmiller (1977), if that term is omitted and the pressure in both phases is assumed to be the same, light particles (bubbles) will be aphysically over-accelerated because the phases will accelerate inversely as their densities for a given pressure gradient. For an air bubble in water this would mean an acceleration of 1000 g. This is compensated for by adding the added-mass term to the momentum equations. This term seems to be important also in the case of liquid-fluidized beds where the densities of the liquid and the particles are of a comparable order of magnitude.

Based on the above considerations, we have undertaken a careful study of available models of multiphase flows, before attempting to solve any single form of the model by numerical means. At present there is a large variety of papers on the mathematical modelling of multiphase flows of rigid or fluid particles dispersed in another fluid. Most of the literature is devoted to the study of the stability of gas- or liquid-fluidized beds (see Homsy, El-Kaissy & Didwania 1980; Liu 1982; Batchelor 1988; Harris & Crighton 1994; Hayakawa, Komatsu & Tsuzuki 1994; Anderson *et al.* 1995). There is far less published on the stability of bubbly flows (Biesheuvel & van Wijngaarden 1984; Biesheuvel & Gorissen 1990; Sangani & Didwania 1993; Lammers & Biesheuvel 1996), and only Jones & Properetti (1985) and Prosperetti & Jones (1987) consider a general model aimed at describing a wide class of multiphase flows.

There are basically two approaches to derive the equations governing the flow of a two-phase mixture (from now on we shall consider only the case of two-phase systems). The first one uses the concept of inter-penetrating continua, considering both phases as two superimposed continua and defining local (time or space) averaged quantities for each of them at each point of the physical space. The mass and momentum conservation equations, in terms of these quantities, are subsequently derived via time or space averaging of the corresponding equations for each phase (see Ishii 1975; Anderson & Jackson 1967; Drew & Segel 1971; Nigmatulin 1979). The resulting equations are not closed and some additional closure relations for the interfacial force between the two phases are to be specified. They are usually constituted on the basis of arguments for the force distribution around a single particle.

The other approach averages the local momentum equations over the entire mixture using an *ensemble averaging* over different realizations of the flow (see Biesheuvel & van Wijngaarden 1984; Batchelor 1988; Biesheuvel & Gorissen 1990; Zhang & Prosperetti 1994). After employing some assumptions for the terms in the resulting averaged equations they end up with a closed set of equations in terms of local averaged quantities again. Although these approaches seems to be very different, they both result in a set of equations that appear to be very similar. These are presented in the next section while discussing two-phase flow models and the relations between them. Apart from the differences in the form of the drag force, elastic reaction to compression and effective diffusivity which are naturally different for solid and fluid particles the only other difference between the existing models is in the formulation of the convective contribution to the local momentum balance.

The large variety of existing models which might be relevant to bubbly flows naturally raises the need for some criterion to distinguish the most adequate among them. In addition to physical arguments, one can also study the properties of the equations. In the present paper we focus on investigating the equations from a mathematical point of view, studying their linear and nonlinear behaviour. In § 3 we present a linear stability analysis of different sets of model equations using a form of the drag force, elastic reaction and viscous dissipation proposed by Biesheuvel & Gorissen (1990). Contrary to the claim by Liu (1982) (in the case of fluidized beds) that the different forms of the relative acceleration between the two phases have only a modest influence upon the linear theory, it was found that in bubbly flow they can significantly alter the result of the linear stability analysis. In § 4 we study the weakly nonlinear hierarchy of equations that follows from the two models having adequate linear behaviour, and derive some necessary conditions for the existence of stable nonlinear waves. The main results of the present work are summarized in § 5.

2. The two-phase flow equations

We shall start the comparative study of the existing models with that proposed by Stuhmiller (1977). Based on the macroscopic equations derived by Ishii (1975) (via time averaging) and using some arguments for the distribution of the forces around a single sphere he derives the following system (all the equations below will be given for the one-dimensional case since some of them are derived for that case only):

$$\frac{\partial \phi}{\partial t} + \frac{\partial(\phi v)}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial(1-\phi)}{\partial t} + \frac{\partial[(1-\phi)u]}{\partial x} = 0, \tag{2.2}$$

$$\phi\rho_2\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) + \phi\rho_1\mu(\phi)\frac{\mathrm{d}(v-u)}{\mathrm{d}t}$$
$$= -\phi\frac{\partial p}{\partial x} - \xi(\phi, |v-u|)\frac{\partial \phi}{\partial x} - \beta(\phi, |v-u|)(v-u) - \phi\rho_2 g, \tag{2.3}$$

$$(1-\phi)\rho_1\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) - \phi\rho_1\mu(\phi)\frac{\mathrm{d}(v-u)}{\mathrm{d}t}$$
$$= -(1-\phi)\frac{\partial p}{\partial x} + \xi(\phi, |v-u|)\frac{\partial \phi}{\partial x} + \beta(\phi, |v-u|)(v-u) - (1-\phi)\rho_1 g, \tag{2.4}$$

with ρ_1, ρ_2 being the densities of the continuous and dispersed phase respectively, u and v their velocities, ϕ the volume fraction of the dispersed phase, $\mathrm{d}(v-u)/\mathrm{d}t = \partial(v-u)/\partial t + v\partial(v-u)/\partial x$ the relative acceleration between the two phases, β a drag parameter, ξ the bulk modulus of elasticity (called dynamic-pressure coefficient by Stuhmiller 1977) and μ the added-mass coefficient. Note that the equations are given here in more generalized form than by Stuhmiller (1977) in order to make the comparison with the other models easier. Note also that in this model there is no diffusion term included. The pressure can easily be eliminated from the two momentum equations to obtain

$$\left(\rho_2\phi + \rho_1\mu(\phi)\frac{\phi}{1-\phi}\right)\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right)$$
$$-\left(\rho_1\mu(\phi)\frac{\phi}{1-\phi} + \rho_1\phi\right)\frac{\partial u}{\partial t} - \left(\phi u + \mu(\phi)\frac{\phi}{1-\phi}v\right)\rho_1\frac{\partial u}{\partial x}$$
$$= -\xi_1(\phi, |v-u|)\frac{\partial \phi}{\partial x} - \beta_1(\phi, |v-u|)(v-u) + \phi(\rho_1-\rho_2)g, \tag{2.5}$$

where $\xi_1(\phi, |v-u|) = \xi(\phi, |v-u|)/(1-\phi), \beta_1(\phi, |v-u|) = \beta(\phi, |v-u|)/(1-\phi)$. That form of momentum balance is more suitable for comparison with other models. Stuhmiller (1977) does not specify the functional dependence of the parameters ξ, μ and β on the void fraction; thus those parameters are still to be specified in a suitable manner, in order to yield a closed model.

A similar generalized momentum equation is derived by Liu (1983) which in terms of the present notations reads

$$\rho_2\phi\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) - \rho_1\phi\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) - \rho_1\frac{\phi}{1-\phi}\mu(\phi)\frac{\mathrm{d}(u-v)}{\mathrm{d}t}$$
$$= -\beta_1(\phi, |v-u|)(v-u) + \phi(\rho_1-\rho_2)g - \frac{\partial p_d}{\partial x} + \frac{\partial}{\partial x}\left(\mu_d\frac{\partial v}{\partial x}\right), \tag{2.6}$$

μ_d being an effective viscosity of the distributed phase which has to be postulated. In order to close the system, the pressure p_d in the distributed phase must also be postulated and then $\partial p_d/\partial x = \rho_2(\partial p_d/\partial(\rho_2\phi))\partial\phi/\partial x = \xi_1(\phi, |v-u|)\partial\phi/\partial x$.

As noted by Liu (1982), the added-mass term has different forms in the different studies depending on the postulated relative particle–fluid acceleration:

$$\frac{\mathrm{d}(u-v)}{\mathrm{d}t} = \left[\frac{\partial}{\partial t} + (u-v)\frac{\partial}{\partial x}\right](u-v) \tag{2.7}$$

or

$$\frac{\mathrm{d}(u-v)}{\mathrm{d}t} = \left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right]u - \left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right]v \tag{2.8}$$

or

$$\frac{\mathrm{d}(u-v)}{\mathrm{d}t} = \left[\frac{\partial}{\partial t} + (u-v)\frac{\partial}{\partial x}\right]u - \left[\frac{\partial}{\partial t} + (v-u)\frac{\partial}{\partial x}\right]v. \tag{2.9}$$

The first two expressions are proposed by Jackson (1971) and the last by Homsy *et al.* (1980). In the model proposed by Stuhmiller (1977), it has yet another, fourth, form:

$$\frac{\mathrm{d}(u-v)}{\mathrm{d}t} = \left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right](u-v). \tag{2.10}$$

In many studies of the stability of fluidized beds, the added-mass force is neglected (see Anderson *et al.* 1995; Harris & Crighton 1994; Hayakawa *et al.* 1994) but this is unreasonable in case of bubbly flows since it is proportional to the liquid density which is much higher than the density of the gas.

In order to close the system one needs to postulate functional relations for the dependence of β, ξ, μ_d and μ on the void fraction ϕ and the relative averaged speed of the particles $|v-u|$.

A reasonable form for the added-mass coefficient is due to Zuber (1964):

$$\mu = \frac{1}{2}\frac{1+2\phi}{1-\phi}. \tag{2.11}$$

It is derived in the case of rigid particles by assuming that each particle moves in a sphere of fluid within a boundary concentric to the particle and representing the influence of other particles, and then calculating its kinetic energy. Biesheuvel & Spoelstra (1989) concluded that it is reliable up to large values of the void fraction in the case of bubbly flows, as well.

Concerning the drag parameter $\beta(\phi, |v-u|)$ most (if not all) of the studies of the stability of two-phase flows use the simple functional form proposed by Needham & Merkin (1983):

$$\beta(\phi) = \frac{D_0}{V_p}\frac{\phi}{(1-\phi)^n}, \tag{2.12}$$

where V_p is the volume of a single particle, D_0 is the Stokes drag and n depends on the Reynolds number of an isolated particle falling at its terminal velocity. It varies from $n=1$ (used by Biesheuvel & Gorissen 1990) to $n=3,4$ (used by Needham & Merkin 1983; Göz 1992; Harris & Crighton 1994), and is often chosen in a way that it is consistent with the experimental correlations of Richardson and Zaki (Richardson 1971) for particle sedimentation in a uniform suspension.

In an extensive study of the drag coefficient C_D of two-phase systems Zuber & Ishii (1979) show that it is a function of the Reynolds number based on the relative velocity of the dispersed phase $|v-u|$ and the mixture viscosity μ_m:

$$Re_m = \frac{2r_d\rho_1|v-u|}{\mu_m} \tag{2.13}$$

where r_d is the particle radius and μ_m (the mixture viscosity) can be approximated as

$$\mu_m = \mu_c\left(1-\frac{\phi}{\phi_{cp}}\right)^{-2.5\phi_{cp}(\mu_d+0.4\mu_c)/(\mu_d+\mu_c)}, \tag{2.14}$$

where ϕ_{cp} is the void fraction of close packing and μ_d, μ_c are the viscosities of the dispersed and continuous phases respectively. For bubbly flows, Zuber & Ishii (1979) take $\phi_{cp}=1$ and hence equation (2.14) reduces $\mu_m=\mu_c(1-\phi)^{-1}$ in the limit of $\mu_d \ll \mu_c$. Only in the case of Stokes flow, however, does the drag coefficient proposed by Zuber & Ishii (1979) give a form for β that is similar to equation (2.12). Since the drag force per unit volume is $3C_D/(8r_d)\rho_1\phi|v-u|(v-u)$, we have $\beta = 3C_D/(8r_d)\rho_1\phi|v-u|$. For the Stokes regime they suggest $C_D = 24/Re_m$ which, when inserted into the above expression for β, gives (2.12) with $n=1$ and a coefficient 0.5. The drag coefficient at higher Reynolds numbers, however, does not match that functional form of (2.12) and involves dependence of β on $|v-u|$ as well. In the undistorted particle regime Zuber & Ishii (1979) suggest that $C_D = 24/Re_m(1+0.1Re_m^{0.75})$. As demonstrated in the next section, if this is used instead of (2.12) the results of the linear stability analysis are changed significantly. At higher Reynolds numbers (distorted particles) the dependence of the drag parameter on $|v-u|$ is even more pronounced, according to the theory of Zuber & Ishii (1979), which will alter the results more.

The third parameter that needs to be for getting a closed model is the bulk modulus of elasticity of the dispersed phase, $\xi(\phi,|v-u|)$. There are a lot of different forms proposed in the fluidized-bed literature for that parameter. Harris & Crighton (1994) used the simple form

$$p_d(\phi) = P\frac{\phi}{\phi-\phi_{cp}}, \tag{2.15}$$

where p_d is 'particle pressure', equivalent to our ξ. Hayakawa *et al.* (1994) used a similar form $\xi(\phi) = P(\phi-\phi_0)^2/(\phi-\phi_{cp})$ while Anderson *et al.* (1995) used the exponential form $p_d = P\phi^3\,\mathrm{e}^{r\phi/(\phi_{cp}-\phi)}$. All these expressions (except the one used by Hayakawa *et al.* (1994)) tend to 0 if $\phi \to 0$ and to ∞ if $\phi \to \phi_{cp}$, i.e. they allow for a stable regime at $\phi > \phi_{cp}$ and correctly vanish at zero particle concentration. They all contain free constants such as (P, r) which allow the critical volume fraction to be adjusted for linear stability of the flow. The particle dynamic viscosity μ_d is usually also postulated in such a way as to vanish at $\phi \to 0$ and increase infinitely as $\phi \to \phi_{cp}$. One such form used by Anderson *et al.* (1995) is $\mu_d = M\phi/(1-(\phi/\phi_{cp})^{1/3})$. M is again a free constant which permits adjustment of the length of the fastest growing mode. In the case of bubbly flows we found only one paper proposing certain forms for ξ and μ_d (Biesheuvel & Gorissen 1990) and it will be discussed below.

Another way to derive the system of equations, adopted by Biesheuvel & van Wijngaarden (1984), Batchelor (1988) and Biesheuvel & Gorissen (1990), is to average the local momentum equation over the entire mixture rather than averaging over each phase separately, i.e. using ensemble averages over a large number of realizations of the system. Under the assumption of small departure from uniformity and zero acceleration of the mixture as a whole, Batchelor (1988) derives the following equations for fluidized beds:

$$\frac{\partial \phi}{\partial t}+\frac{\partial (v\phi)}{\partial x}=0, \tag{2.16}$$

$$\phi(1+\theta)\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}\right)-\phi\zeta v\frac{\partial v}{\partial x}=-Q\frac{\partial \phi}{\partial x}+\frac{\partial}{\partial x}\left(\phi\eta\frac{\partial v}{\partial x}\right)-\frac{\gamma\overline{g}}{U}\phi(v-U), \tag{2.17}$$

where $U(\phi)$ is the mean velocity of the particles in a homogeneous dispersion when the particles move only under the action of gravity. γ depends on the Reynolds number of the flow and takes values between 1 (Stokes) and 2 (high Reynolds numbers), $\theta=\rho_1/\rho_2\mu(\phi)$, $\zeta(\phi)=\rho_1/\rho_2\phi\partial\mu(\phi)/\partial\phi$, $\overline{g}=g(\rho_2-\rho_1)/\rho_2$, Q is the bulk modulus of elasticity and η is an effective diffusion coefficient.

Biesheuvel & Gorissen (1990) derive in a similar manner another momentum equation:

$$\frac{\partial\{\phi[\rho_2 v+\rho_1\mu(v-U_m)]\}}{\partial t}+\frac{\partial\{\phi[\rho_2 v+\rho_1\mu(v-U_m)]v\}}{\partial x}-\phi\rho_1\frac{\partial U_m}{\partial t}$$
$$=-\frac{9\mu_c}{r_d^2}\frac{\phi}{(1-\phi)^2}\left[(v-U_m)+\frac{\delta_e}{\phi}\frac{\partial \phi}{\partial x}\right]-\phi(\rho_2-\rho_1)g+\frac{\partial}{\partial x}\left(-p_e+\mu_d\frac{\partial v}{\partial x}\right). \tag{2.18}$$

Under the assumption of zero mixture acceleration, however, and if we write the equation in a framework related to the mixture ($U_m=0$, U_m is the mixture velocity), we can manipulate the convective part using the continuity equations in order to produce the same equation as the one derived by Batchelor (1988). Note that the right-hand side of the momentum equation of Batchelor, for $\gamma=1$, can be reduced to the right-hand side of equation (2.18) of Biesheuvel & Gorissen (1990). If $\partial U_m/\partial t=0$, $U_m=\phi v+(1-\phi)u$, one can exclude the continuous-phase velocity u from the momentum equation (2.6) and get an equation similar to (2.17). If $\rho_1/\rho_2\ll 1$ (gas-fluidized beds) the resulting equation will be the same (up to the specification of ξ,β and μ_d). In the case of liquid-fluidized beds or bubbly flows the convection parts of those momentum equations differ significantly. The equation derived by Batchelor (1988) is deduced under the assumption of small deviation from uniformity and is an essentially one-dimensional equation. That is why it cannot be used beyond the one-dimensional linear and nonlinear analysis of two-phase flows. In our further analysis we shall use only the momentum equations (2.6) and (2.18). Since the expressions for p_d and μ_d discussed above concern only the case of fluidized beds and are not relevant to bubbly flows we shall adopt the form of these terms proposed by Biesheuvel & Gorissen (1990) (actually their derivation is based on the analysis of Batchelor 1988):

$$\left.\begin{aligned}&\frac{\partial p_d}{\partial \phi}=\frac{\partial p_e}{\partial \phi}+\frac{9\mu_c}{r_d^2}\frac{\delta_e}{1-\phi},\quad p_e=\phi(\rho_2+\rho_1\mu)H(\phi)v_0^2(\phi),\\&\delta_e=Pr_d v_o(\phi)(H(\phi))^{1/2},\quad H(\phi)=\frac{\phi}{\phi_{cp}}\left(1-\frac{\phi}{\phi_{cp}}\right),\\&\mu_d=M\phi(\rho_2+\rho_1\mu(\phi))r_d v_0(\phi)(H(\phi))^{1/2},\end{aligned}\right\} \tag{2.19}$$

where $v_0(\phi)$ is the mean velocity of rise of a uniform suspension in a stagnant liquid and $\phi_{cp}=0.61$. Here, the dispersed-phase pressure p_e includes only the so-called 'kinetic' contribution to it (in analogy to the pressure in pure liquids, derived from statistical mechanics principles). Sangani & Didwania (1993) also derived an analytic expression for the 'potential' contribution to the pressure for some simple periodic configurations of the bubbles in the flow which is proportional to the square of the relative velocity $V = v - U_m$:

$$p_e^p = \lambda_1 \rho_1 \phi^2 \frac{V^2}{(1-\phi)^2}. \tag{2.20}$$

Here λ_1 is a constant depending on the configuration of the particles: $\lambda_1 = 0.386$ for a simple cubic array, $\lambda_1 = -1.26$ for a body-centred array, and $\lambda_1 = -1.17$ for a face-centred cubic array. Both forms of the dispersed-phase pressure (with and without the potential contribution) are used in the linear analysis below. Equation (2.20) is valid, however, only for some unrealistic arrangements of the bubbles and in the case of a random bubble distribution no analytic expression can be derived. That is why the linear stability results have merely a qualitative value and in the nonlinear stability analysis only the 'kinetic' contribution is accounted for.

To summarize, a number of two-phase flow models proposed in the literature can be presented in a general form (the momentum equation is obtained by excluding the pressure from the momentum equations for each of the phases which is possible only in the one-dimensional case; in the multi-dimensional case the system will contain two momentum equations):

$$\begin{aligned} A_1(\phi)\frac{\partial v}{\partial t} + A_2(\phi)\frac{\partial u}{\partial t} + B_1(\phi)v\frac{\partial v}{\partial x} + B_2(\phi)u\frac{\partial u}{\partial x} + B_3(\phi)u\frac{\partial v}{\partial x} + B_4(\phi)v\frac{\partial u}{\partial x} \\ = -\beta(\phi, |v-u|)(v-u) - \xi(\phi)\frac{\partial \phi}{\partial x} + \phi(\rho_1 - \rho_2)g + \frac{\partial}{\partial x}\left(\mu_d(\phi)\frac{\partial v}{\partial x}\right), \end{aligned} \tag{2.21}$$

$$\frac{\partial \phi}{\partial t} + \frac{\partial(\phi v)}{\partial x} = 0, \tag{2.22}$$

$$\frac{\partial(1-\phi)}{\partial t} + \frac{\partial[(1-\phi)u]}{\partial x} = 0. \tag{2.23}$$

From now on we drop the subscript on the drag parameter and the bulk modulus of elasticity and suppose that ξ depends on ϕ solely. That is the case with β in most of the cases discussed above, as well. The coefficients A_1, A_2, B_1 to B_4 depend on the forms for the added-mass term postulated by the different authors. That force is negligible in case of gas-fluidized beds (more precisely if $\rho_1/\rho_2 \ll 1$) but is evidently important for bubbly flows. Since it is difficult, based only on physical arguments, to justify the use of any of the proposed momentum equations we study below the linear stability of the different models and use the results to find the applicability of some of them for modelling of bubbly flows.

A similar general class of models has been proposed by Jones & Prosperetti (1985) (including only first derivatives of the flow variables) and Prosperetti & Jones (1987) (including also second derivatives of the velocity components). These models are sufficiently broad to accommodate a wide variety of physical phenomena such as surface tension, correlation effects arising from the averaging of the conservation equations, added mass and dissipation. It is clear that (2.21)–(2.23) can be cast into the form proposed by Prosperetti & Jones (1987) and, as expected, most of the qualitative results of the linear analysis, presented in the next section, agree well with the conclusions of Jones & Prosperetti (1985) and Prosperetti & Jones (1987). We also examined the effect of different forms of the added-mass terms, nonlinear relations for the drag and different forms of the pressures in the two phases. In addition, we carry out a weakly nonlinear analysis, using the most credible forms for the different terms (available in the literature and 'approved' by the linear analysis).

3. Linear stability analysis

Many features of the physical system under consideration (fluidized beds, bubbly flows) can be revealed through one-dimensional linear stability analysis of the equations. Equations (2.21) to (2.23) accept a trivial solution representing a uniform dispersion:

$$\phi = \phi_0, \quad u = 0, \quad v = v_0 = \frac{\phi_0(\rho_1 - \rho_2)g}{\beta(\phi_0)}. \tag{3.1}$$

Here we consider the physical situation of lighter dispersed phase in a stagnant liquid. If we further suppose that an infinitesimally small disturbance is imposed onto the flow we can study its effect, up to the first order with respect to its amplitude, by linearizing the governing equations. The linearized equations corresponding to the basic solution (3.1) read

$$\frac{\partial \phi}{\partial t} + \phi_0 \frac{\partial v}{\partial x} + v_0 \frac{\partial \phi}{\partial x} = 0, \tag{3.2}$$

$$-\frac{\partial \phi}{\partial t} + (1 - \phi_0)\frac{\partial u}{\partial x} = 0, \tag{3.3}$$

$$\begin{aligned} pcA_1(\phi_0)\frac{\partial v}{\partial t} + A_2(\phi_0)\frac{\partial u}{\partial t} + B_1(\phi_0)v_0\frac{\partial v}{\partial x} + B_4(\phi_0)v_0\frac{\partial u}{\partial x} \\ = -\beta(\phi_0)(v-u) - \beta'(\phi_0)v_0\phi - \xi(\phi_0)\frac{\partial \phi}{\partial x} + \phi(\rho_1 - \rho_2)g + \mu_d(\phi_0)\frac{\partial^2 v}{\partial x}. \end{aligned} \tag{3.4}$$

Here and below $'$ denotes differentiation with respect to ϕ. After differentiation of (3.4) with respect to x and substitution of $\partial v/\partial x$ and $\partial u/\partial x$ from (3.2) and (3.3) one gets

$$\begin{aligned} &\left(\frac{B_4 v_0}{1-\phi_0} - \frac{B_1 v_0}{\phi_0} - \frac{A_1 v_0}{\phi_0}\right)\frac{\partial^2 \phi}{\partial t \partial x} + \left(\xi_0 - B_1\frac{v_0^2}{\phi_0}\right)\frac{\partial^2 \phi}{\partial x} + \left(\frac{A_2}{1-\phi_0} - \frac{A_1}{\phi_0}\right)\frac{\partial^2 \phi}{\partial t} \\ &-\beta_0\left(\frac{1}{\phi_0} + \frac{1}{1-\phi_0}\right)\frac{\partial \phi}{\partial t} + \left(\beta_0' v_0 + (\rho_2 - \rho_1)g - \frac{\beta_0 v_0}{\phi_0}\right)\frac{\partial \phi}{\partial x} \\ &+\frac{\mu_d v_0}{\phi_0}\frac{\partial^3 \phi}{\partial x^3} + \frac{\mu_d}{\phi_0}\frac{\partial^3 \phi}{\partial t \partial x^2} = 0. \end{aligned} \tag{3.5}$$

All the coefficients in that equation are evaluated at the basic void fraction ϕ_0. If we further seek a solution in the form $\phi(x,t) \approx e^{\sigma_r t}e^{[i(kx+\sigma_i t)]}$ and substitute it into (3.5) we shall get the dispersion relation which in this case can be presented as

$$a\sigma^2 + (b + ci)\sigma + d + ei = 0; \tag{3.6}$$

a, b, c, d, e are real quantities, which depend on the wavenumber k and the coefficients of the equation (3.5) and $\sigma = \sigma_r + i\sigma_i$. The flow is now linearly unstable if $\sigma_r > 0$ and the disturbance will grow exponentially in time. Since a and b always have the same sign (for all models A_2 is negative and A_1 positive) we can rewrite that condition as

$$\sqrt{(b^2 - c^2 - 4ad)^2 + (2bc - 4ae)^2} \geqslant c^2 + 4ad + b^2. \tag{3.7}$$

If $c^2 + 4ad + b^2 < 0$ that condition is always satisfied. For some of the models (e.g. Biesheuvel & Gorissen 1990) $c^2 + 4ad + b^2 > 0$ but in some cases it is (in principle) possible for that expression to have a negative sign. Our investigation for air bubbles in water showed that for all the models we considered this sufficient condition for instability is weaker than another condition that follows now. If we assume that $c^2 + 4ad + b^2 > 0$ we can further derive that $\sigma_r > 0$ if

$$b^2 d - ae^2 + bce < 0. \tag{3.8}$$

From this condition we can recognize the importance of the added-mass force and the elastic reaction for maintaining the stability of the equations. If we suppose that $\xi(\phi) = \mu_d(\phi) = \mu(\phi) = 0$ the condition (3.8) for the general system (2.21)–(2.23) reads†

$$\begin{aligned}
&-\beta^2 B_1 v_0^2(1-\phi_0)k^2 - [\beta v_0(1-\phi_0) - \beta' v_0(1-\phi_0)\phi_0 + \rho_1 g\phi_0(1-\phi_0) - g\rho_2\phi_0(1-\phi_0)]^2 \\
&\quad \times (A_1(1-\phi_0) - A_2\phi_0)k^2 + \beta[A_1 v_0(1-\phi_0) + B_1 v_0(1-\phi_0)] \\
&\quad \times [\beta v_0(1-\phi_0) - \beta' v_0\phi_0(1-\phi_0) + g\rho_1\phi_0(1-\phi_0) - g\rho_2\phi_0(1-\phi_0)]k^2 < 0.
\end{aligned} \tag{3.9}$$

For all of the models discussed above (Stuhmiller 1977; Jackson 1971; Homsy *et al.* 1980; Biesheuvel & Gorissen 1990; Batchelor 1988; Hayakawa *et al.* 1994; Harris & Crighton 1994; Anderson *et al.* 1995) $A_1 = B_1 > 0$ and $A_2 < 0$. Since A_1 is always of the form $\rho_2\phi + \rho_1\mu(\phi)$, and if we additionally suppose that $\rho_2 \ll \rho_1$ (the case of bubbly flows), it follows that the second condition for instability is always satisfied.

Next we study the influence of different terms in the momentum equations on the linear stability of the system. Since, for physically realistic values of the different parameters, the equations are convection-dominated we start by evaluating different forms of the convective terms. The two most recent studies (Biesheuvel & Gorissen 1990 and Sangani & Didwania 1993) derive those terms from conservation of the Kelvin impulse:

$$\frac{\mathrm{D}I}{\mathrm{D}t} = 0, \tag{3.10}$$

and they result in the same form of the convective contribution to the momentum equations. However, this form is quite complicated and inconvenient for an eventual numerical treatment. Therefore, in an attempt to justify some of the models derived by spatial or temporal averaging of the momentum equations of both phases, we compare their linear stability behaviour to the behaviour of the model derived by Biesheuvel & Gorissen (1990)(case (i)). The other terms in the equations are used in the form proposed by Biesheuvel & Gorissen (1990): (2.19) for $\xi(\phi)$ and $\mu_d(\phi)$ and $\beta = D_0/V_p\phi/(1-\phi)$. The two other forms of the convective contribution that are used are the ones derived by Stuhmiller (1977) (2.10), case (ii) and Jackson (1971) (2.8), case (iii). The other forms suggested by Jackson (1971) (2.7) and Homsy *et al.* (1980) (2.9) produce quite erroneous linear stability results and they are not further discussed in the paper. The constants P and M in (2.19) are fixed to 1.

In figure 1 we present the neutral stability curves and the growth rates as function of the wavenumber k obtained using values of the parameters corresponding to air bubbles with a radius of 0.4 mm in water. As should be expected (see also Jones & Prosperetti 1985), if the effective diffusivity is $\mu_d = 0$ the neutral stability curves would be straight lines parallel to the y-axis because then all the terms in the stability criterion (3.8) contain k^2 only and it can be divided by k^2 (for $k \neq 0$). Moreover, μ_d plays a purely dispersive role in the linear stability analysis and it does not alter the value of the critical void fraction significantly over a reasonable range of wavenumbers.

† All analytical calculations in this study have been performed with the aid of the computer algebra code Maple, a trademark of Waterloo Maple Software and the University of Waterloo.

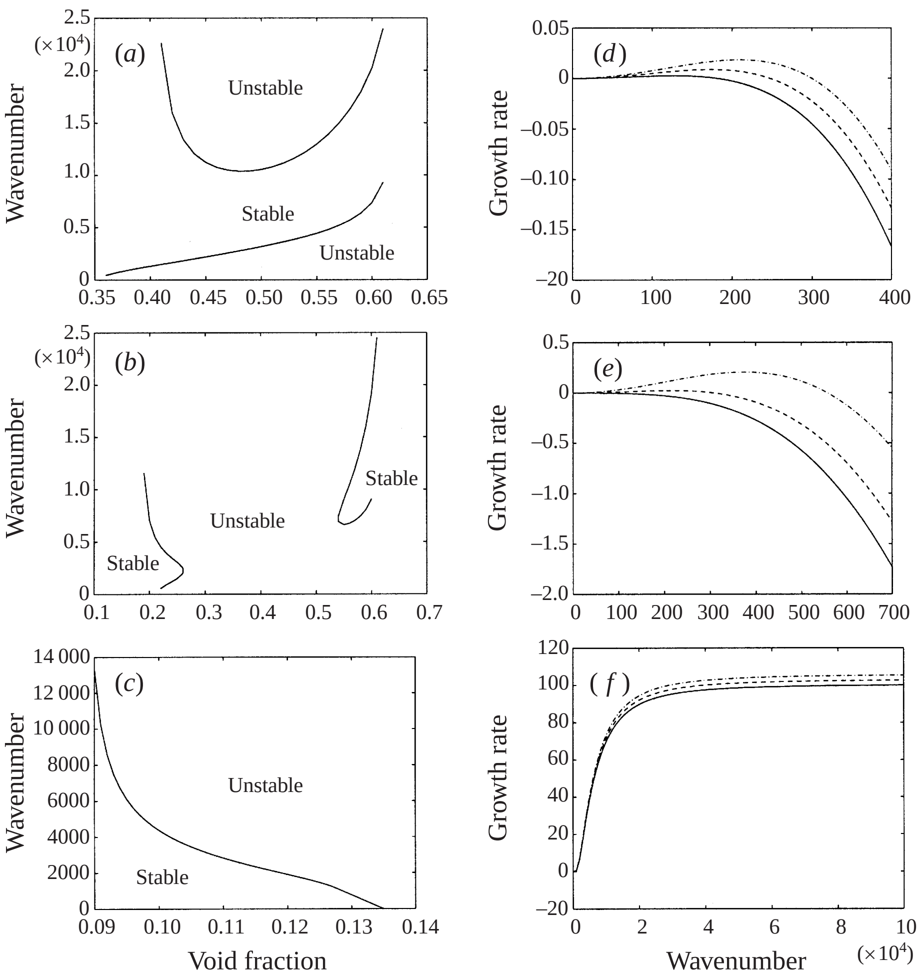

FIGURE 1. Neutral stability curves (left): (*a*) case (i), (*b*) case (ii), (*c*) case (iii) and growth rate vs. k (right): (*d*) case (i): ——, $\phi = 0.353$; - - - -, $\phi = 0.354$; —·—·—, $\phi = 0.355$; (*e*) case (ii): ——, $\phi = 0.212$; - - - -, $\phi = 0.215$; —·—·—, $\phi = 0.22$; (*e*) case (iii): ——, $\phi = 0.13$, - - - -, $\phi = 0.131$; —·—·—, $\phi = 0.132$.

It is clear from figure 1 that the different forms of the convection terms give quite different linear stability results. This is in contradiction with what Liu (1983) has noted in case of fluidized beds. It is due to the fact that the added-mass force (which actually changes the formulation of the convection terms) has a much more pronounced contribution to the total force balance in the former case. It is proportional to ρ_1 and in that case $\rho_2 \ll \rho_1$. In an experimental study Matuszkiewicz, Flamand & Bouré (1987) found that a uniform bubbly flow can undergo a transition at values of the void fraction larger than 0.25. Since the expressions for δ_e and μ_d contain free parameters of order of 1, however, this quantitative criterion is not decisive. But it is clear that qualitatively reasonable results are obtained with all the models (i), (ii) and (iii). They predict critical void fractions 0.222, 0.13 and 0.353 respectively. A more clear distinction between the other three models can be made if we study the dependence of the growth rate σ_r on the wavenumber k for values of the void fraction close to the critical. Since it is reasonable to accept that the frequency of

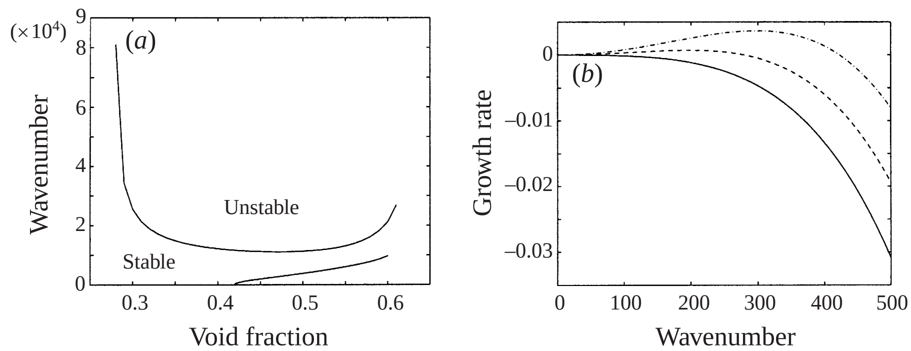

FIGURE 2. Case (i) with a modified expression for the drag proposed by Zuber & Ishii (1979). (*a*) Neutral stability curves and (*b*) growth rate vs. k: ——, $\phi = 0.418$; - - - -, $\phi = 0.419$; —·—·—, $\phi = 0.42$.

the fastest growing mode will be close (as an order of magnitude) to the frequency of the (generally nonlinear) waves measured in the experiments we can use some of the conclusions of Matuszkiewicz *et al.* (1987). They observed that it increases with increasing value of the void fraction. Moreover that frequency is very low – less than 10 Hz. In the long-wave limit ($k \to 0$) we get from (3.6) that σ is always real (a and b are real) and hence $\sigma_i = 0$. Thus, the wavenumber k is at most of the order of the frequency σ_i. From the figures corresponding to (iii) we can conclude that this model is not relevant to the qualitative experimental data because, first, the wavenumber (respectively the frequency) of the fastest growing mode is very high and, second, it is practically independent of the the void fraction. Thus, the only model (beside the one of Biesheuvel & Gorissen 1990) relevant to bubbly flows is that of Stuhmiller (1977).

In order to examine the influence of the formulation of the drag force we have used (2.18) with a drag coefficient proposed by Zuber & Ishii (1979): $C_D = 24/Re_m(1 + 0.1Re^{0.75})$ which is relevant to non-distorted particles at intermediate Reynolds numbers. The rest of the momentum equation is kept the same as in the model formulated by Biesheuvel & Gorissen (1990). The neutral stability curves and the growth rate (as function of k) are presented in figure 2. The modified drag coefficient stabilizes the flow with respect to low-wavenumber disturbances, raising the critical void fraction to 0.418. On the other hand, it plays a destabilizing role at high wavenumbers and its stability behaviour generally resembles the one of model (iii) (see figure 1). It is difficult to judge the response of the model to high-wavenumber disturbances because no relevant experimental data exist but the model is obviously overstabilized at low wavenumbers.

Another important term in the equations which influences the stability of the model is the dispersed-phase pressure. In the results above we used only the 'kinetic' contribution to it in the form suggested by Biesheuvel & Gorissen (1990) (equation (2.19)). If we include also the 'potential' contribution in the form calculated by Sangani & Didwania (1993) (equation (2.20)) the model is stabilized if λ_1 is positive and destabilized if it is negative (as predicted by these authors). The results for the neutral stability curves for $\lambda_1 = 0.386$ and $\lambda_1 = -1.17$ are presented in figure 3. The stabilization in the case of a simple cubic array configuration of the bubbles is relatively slight and the critical void fraction increases to 0.388. The negative 'potential' pressure in the case of the face-centred cubic array, however, significantly destabilizes the equations. They are unstable for all values of the void fraction larger than 0.035 (and large enough k). It also changes the qualitative linear behaviour

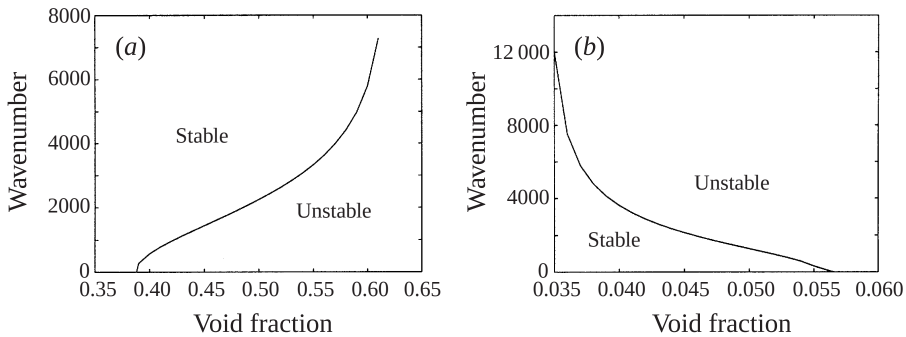

FIGURE 3. Neutral stability curves in case of non-zero 'potential' contribution to the pressure calculated by Sangani & Didwania (1993); (*a*) $\lambda_1 = 0.386$ and (*b*) $\lambda_1 = -1.17$.

of the model, making the critical wavenumber decrease with the increase of the void fraction. Since the experimental evidence shows that the flow is stable to much higher void fraction values it seems that this expression for the 'potential' pressure contribution has a very limited applicability. As already mentioned, in the case of a random (and more realistic) arrangement of the bubbles in the suspension, there is no analytic expression for this contribution and we do not consider it in the nonlinear analysis below.

It appears that the results of the linear stability analysis alone cannot decisively demonstrate the applicability of any single model. It is useful, however, in sorting out models that are clearly on the wrong track from those that retain some relevance in describing certain aspects of the flow physics.

4. Weakly nonlinear analysis

In this section, we consider only the models (i) and (ii), which were not ruled out by the linear stability analysis.

Since it is reasonable to assume that, at least close to the critical value of the void fraction, the fastest growing mode corresponds to very low wavenumber we can expand σ in series of k around $k = 0$:

$$\sigma = -\mathrm{i}c_0 k + \delta k^2 + \mathrm{i}Fk^3 + O(k^4), \tag{4.1}$$

where $c_0 = e_k/b|_{k=0}$, the subscript k denoting a differentiation with respect to k, and

$$\left.\begin{aligned} \delta &= \left(\frac{e_k}{b}\right)^2 \frac{a}{b} - \frac{c_k e_k}{b^2} - \frac{d_{kk}}{2b}\bigg|_{k=0}, \\ F &= \frac{c_k^2 e_k}{b^3} - \frac{3ac_k e_k^2}{b^4} + \frac{2a^2 e_k^3}{b^5} + \frac{c_k d_{kk}}{b^2} - \frac{ae_k d_{kk}}{b^3}\bigg|_{k=0}. \end{aligned}\right\} \tag{4.2}$$

Our calculation of δ according to (4.2) gives for case (i)

$$\begin{aligned} \delta &= c_0^2 \frac{\rho_1\mu_0\phi_0(1-\phi_0) + \rho_2\phi_0(1-\phi_0)}{\beta_0} \\ &\quad - c_0 \frac{2\rho_2 v_0\phi_0(1-\phi_0) + 2\rho_1\mu_0 v_0\phi_0(1-\phi_0) - \mu_0'\rho_1 v_0\phi_0^2(1-\phi_0)}{\beta_0} \\ &\quad - \frac{\xi_0\phi_0(1-\phi_0) - \rho_2 v_0^2\phi_0(1-\phi_0) - \rho_1\mu_0 v_0^2\phi_0(1-\phi_0) + \rho_1\mu_0' v_0^2\phi_0^2(1-\phi_0)^2}{\beta_0} \end{aligned} \tag{4.3}$$

and for case (ii)

$$\delta = c_0^2 \frac{\rho_1\mu_0\phi_0 + \rho_1\phi_0^2(1-\phi_0) + \rho_2\phi_0(1-\phi_0)^2}{\beta_0(1-\phi_0)}$$
$$-c_0 \frac{2\rho_2 v_0\phi_0(1-\phi_0)^2 + 2\rho_1 v_0\mu_0\phi_0 - \rho_1 v_0\mu_0\phi_0^2}{\beta_0(1-\phi_0)}$$
$$-\frac{\xi_0(1-\phi_0)\phi_0 - \rho_2 v_0^2\phi_0(1-\phi_0) - \rho_1 v_0^2\mu_0\phi_0}{\beta_0}. \tag{4.4}$$

It is clear from (4.1) that the fastest growing mode should correspond to k which is of order $\sqrt{\delta}$, which means that if k is small δ is also a small parameter. Calculations for δ around the critical value of the void fraction show that in case (i) it is equal to 0.108×10^{-2} for air bubbles of radius 0.4 mm in water and in case (ii) it is equal to 0.61×10^{-2}. Following Hayakawa *et al.* (1994) we use $\epsilon = \sqrt{\delta}$ as a small parameter for a weakly nonlinear analysis of those models. First, we scale the space and time coordinates as

$$\tau = \epsilon^3 t, \quad \eta = \epsilon(x - c_0 t), \tag{4.5}$$

where c_0 is the first approximation to the propagation velocity of the linear waves. The same scaling is used also by Hayakawa *et al.* (1994) (Gardner–Morikawa transformation) in their analysis of fluidized beds. Then we expand the two velocities and the void fraction in series of ϵ :

$$\left.\begin{aligned} u &= u_2\epsilon^2 + u_3\epsilon^3 + \cdots, \\ v &= v_0 + v_2\epsilon^2 + v_3\epsilon^3 + \cdots, \\ \phi &= \phi_0 + \phi_2\epsilon^2 + \phi_3\epsilon^3 + \cdots. \end{aligned}\right\} \tag{4.6}$$

The parameters $\beta(\phi)$, $\xi(\phi)$ and $\mu_d(\phi)$ can also be expanded in series of ϵ. But first we have to mention that, as can be seen from (4.4) and (4.3), the small parameter ϵ is fixed if the properties of the fluid as well as ϕ_0, v_0, β_0, μ_0 and ξ_0 are fixed. In order to use ϵ as a perturbation parameter we therefore need to assume that at least one of the other parameters is dependent on ϵ. The most convenient choice is ξ_0 which can be written as

$$\xi_0 = \xi_0^0 + \xi_0^1\epsilon^2, \tag{4.7}$$

ξ_0^0 and ξ_0^1 being independent of ϵ. The corresponding expressions for them are derived from (4.4) or (4.3). Then we need to use a multivariate Taylor expansion for $\xi(\phi, \epsilon)$. It is clear from (4.7), however, that only the zeroth and second derivatives of ξ with respect to ϵ will contribute to the expansion, being equal to ξ_0^0 and ξ_0^1 respectively. The Taylor expansion then reads

$$\xi(\phi, \epsilon) = \xi_0^0 + (\xi_0'\phi_2 + \xi_0^1)\epsilon^2 + \xi_0'\phi_3\epsilon^3 + (\xi_0'\phi_4 + \xi_0''\phi_2^2/2)\epsilon^4 + O(\epsilon^5), \tag{4.8}$$

where

$$\xi_0' = \left.\frac{\partial \xi}{\partial \phi}\right|_{\phi=\phi_0, \epsilon=0}, \tag{4.9}$$

$$\xi_0'' = \left.\frac{\partial^2 \xi}{\partial \phi^2}\right|_{\phi=\phi_0, \epsilon=0}, \tag{4.10}$$

The expansions for β and μ_d read

$$\zeta(\phi) = \zeta_0 + \zeta_0'\phi_2\epsilon^2 + \zeta_0'\phi_3\epsilon^3 + (\zeta_0'\phi_4 + \zeta_0''\phi_2^2/2)\epsilon^4 + O(\epsilon^5), \tag{4.11}$$

where ζ is β or μ_d and subscript 0 means that the corresponding quantity is evaluated at void fraction ϕ_0.

After a substitution of (4.5), (4.6), (4.8) and (4.11) in the models (i) and (ii) we get a set of continuity and momentum equations in series of powers of ϵ. Up to $O(\epsilon^3)$ the relations are linear and reproduce the results of the linear analysis given above in the long-wave limit. Of particular interest for the approximations below are the continuity equations for ϵ^2 and ϵ^3:

$$\left.\begin{array}{ll} u_2 = -\dfrac{c_0}{1-\phi_0}\phi_2, & v_2 = \dfrac{c_0 - v_0}{\phi_0}\phi_2, \\ u_3 = -\dfrac{c_0}{1-\epsilon_0}\phi_3, & v_3 = \dfrac{c_0 - v_0}{\phi_0}\phi_3. \end{array}\right\} \tag{4.12}$$

If we further use the continuity equations for ϵ^5 and substitute them in the momentum equation for ϵ^4, using also (4.12), we get a Korteweg–de Vries equation for ϕ_2:

$$\frac{\partial\phi_2}{\partial\tau} + \lambda_0\frac{\partial\phi_2^2}{\partial\eta} + \gamma_0\frac{\partial^3\phi_2}{\partial\eta^3} = 0, \tag{4.13}$$

with

$$\left.\begin{array}{l} \lambda_0 = \dfrac{(c_0 - v_0)(1-\phi_0)}{\phi_0} - \dfrac{c_0\phi_0}{1-\phi_0} - 1/2v_0\dfrac{\beta_0''}{\beta_0}(1-\phi_0)\phi_0 - \dfrac{\beta_0'}{\beta_0}[c_0 - v_0(1-\phi_0)], \\ \gamma_0 = \mu_{d,0}\dfrac{(c_0 - v_0)(1-\phi_0)}{\beta_0}. \end{array}\right\} \tag{4.14}$$

It is interesting to note that those coefficients are the same for both (i) and (ii) models, i.e. they do not depend on the formulation of the inertial forces. It shows that the transform (4.5) and the expansion (4.8) normalize both models on the critical point of linear instability and qualitatively the models behave (up to higher-order effects) identically in the vicinity of their critical points. The added-mass term appears to play a role in the equation for the next approximation ϕ_3. After a substitution of β_0, c_0 and v_0 in (4.14), using the expression (2.19) and some manipulating we get that

$$\gamma_0 = 2\frac{\mu_{d,0}ga^4(\rho_1 - \rho_2)(1-\phi_0)^2(\phi_0 - 1)}{81\mu_l^2} < 0, \tag{4.15}$$

and

$$\lambda_0 = \frac{g(\rho_1 - \rho_2)(3\phi_0 - 2)}{\beta_0} < 0, \tag{4.16}$$

for $0 < \phi_0 < \phi_{cp}$. The same Korteweg–de Vries equation for ϕ_2, for fluidized beds, is derived by Harris & Crighton (1994) and Hayakawa *et al.* (1994) with the coefficients λ_0, γ_0 being negative and positive respectively. It is a well known fact that the Korteweg–de Vries equation is integrable in a closed form and this particular solution is a single soliton:

$$\phi_2(\eta) = \frac{6\gamma_0}{\lambda_0}k^2\mathrm{sech}^2[k(\eta - \eta_0 - 4\gamma_0 k^2\tau)]. \tag{4.17}$$

Thus, for bubbly flows the Korteweg–de Vries soliton will propagate as a wave of a higher void fraction than the uniform value ϕ_0. That corresponds to some extent to the experimental evidence that (if the test section is sufficiently long and a cylindrical tube is used) a region of high void fraction of bubbles is usually formed at a certain height of the dispersion which causes the bubbles to coalesce and form large slug bubbles.

If we further use the momentum equation for ϵ^5 and the continuity equations for ϵ^6 and substitute the expressions for v_4 and u_4 from the continuity equations for ϵ^4 we finally end up with the following equation for ϕ_3:

$$\frac{\partial \phi_3}{\partial \tau} + 2\lambda_0 \frac{\partial \phi_2 \phi_3}{\partial \eta} + \gamma_0 \frac{\partial^3 \phi_3}{\partial \eta^3} = -\frac{\partial^2 \phi_2}{\partial \eta^2} - \lambda_0' \frac{\partial^2 \phi_2^2}{\partial \eta^2} - \gamma_0' \frac{\partial^4 \phi_2}{\partial \eta^4}. \tag{4.18}$$

In that derivation the equation (4.13) has also been used. The coefficients λ_0' and γ_0' depend on the model that has been used and are given in Appendix A for both models (i) and (ii). Using higher-order approximation for the momentum and continuity equations the next equation of the hierarchy can be derived of the form:

$$\frac{\partial \phi_4}{\partial \tau} + 2\lambda_0 \frac{\partial \phi_2 \phi_4}{\partial \eta} + \gamma_0 \frac{\partial^3 \phi_4}{\partial \eta^3} = f\left(\phi_2, \phi_3, \frac{\partial \phi_2}{\partial \eta}, \frac{\partial \phi_3}{\partial \eta}, ..., \frac{\partial^5 \phi_2}{\partial \eta^5}\right). \tag{4.19}$$

Hayakawa *et al.* (1994), combining (4.13) and (4.18), derived an equation for $\psi = \phi_2 + \epsilon\phi_3$ instead of (4.18):

$$\frac{\partial \psi}{\partial \tau} + \lambda_0 \frac{\partial \psi^2}{\partial \eta} + \gamma_0 \frac{\partial^3 \psi}{\partial \eta^3} = -\epsilon\left(\frac{\partial^2 \psi}{\partial \eta^2} + \lambda_0' \frac{\partial^2 \psi^2}{\partial \eta^2} + \gamma_0' \frac{\partial^4 \psi}{\partial \eta^4}\right) + O(\epsilon^2). \tag{4.20}$$

The equation hierarchy (4.13)–(4.18), however, can be solved analytically because the only nonlinear equation of the hierarchy is the first one, which has a closed form solution. All the next approximations satisfy linear non-homogeneous equations of the type (4.19). That equation hierarchy is similar to the one derived by Harris & Crighton (1994) except for the right-hand side of (4.18).

In an attempt to find a permanent-wave solution to (4.18) we first change the variables:

$$\theta = k(\eta - \eta_0 - 4\gamma_0 k^2 \tau), \quad s = \tau, \tag{4.21}$$

as used by Harris & Crighton (1994). Then it becomes

$$\frac{\partial \phi_3}{\partial s} - 4\gamma_0 k^3 \frac{\partial \phi_3}{\partial \theta} + 2\lambda_0 k \frac{\partial \phi_2 \phi_3}{\partial \theta} + \gamma_0 k^3 \frac{\partial^3 \phi_3}{\partial \theta^3} = -k^2 \frac{\partial \phi_2}{\partial \theta^2} - k^2 \lambda_0' \frac{\partial^2 \phi_2^2}{\partial \theta^2} - k^4 \gamma_0' \frac{\partial^4 \phi_2}{\partial \theta^4}. \tag{4.22}$$

Using the solution (4.17) of (4.13) we can try to find a permanent-wave solution to (4.22) which means $\partial \phi_3 / \partial s = 0$. Note that (4.17) is also a permanent solution in the same moving frame (4.21). Equation (4.22) can be integrated once to obtain

$$-4\gamma_0 k^2 \phi_3 + 2\lambda_0 \phi_2 \phi_3 + \gamma_0 k^2 \frac{\partial^2 \phi_3}{\partial \theta^2} = -k \frac{\partial \phi_2}{\partial \theta} - k\lambda_0' \frac{\partial \phi_2^2}{\partial \theta} - k^3 \gamma_0' \frac{\partial^3 \phi_2}{\partial \theta^3} + A. \tag{4.23}$$

Substituting (4.17) into (4.23) we get

$$\begin{aligned}-4\phi_3 + 12\,\mathrm{sech}^2\,\theta\phi_3 + \frac{\partial^2 \phi_3}{\partial \theta^2} = \frac{12k}{\lambda_0}(&\tanh\theta\,\mathrm{sech}^2\,\theta + \frac{12\lambda_0'\gamma_0 k^3}{\lambda_0}\tanh\theta\,\mathrm{sech}^4\,\theta \\ &+4k^2\gamma_0'\tanh\theta\,\mathrm{sech}^2\,\theta - 12k^2\gamma_0'\tanh\theta\,\mathrm{sech}^4\,\theta) + A. \end{aligned} \tag{4.24}$$

One solution to the homogeneous equation is given by $w = \mathrm{sech}^2\,\theta\tanh\theta$. Then (4.24) can be integrated by substituting ϕ_3 of the form $\phi_3 = wv$ and obtaining an integrable equation for v. If we denote the right-hand side of (4.24) by $R(\theta)$, v reads

$$v = B + C\int \frac{\mathrm{d}\theta}{w^2(\theta)} + \int \frac{\mathrm{d}\theta}{w^2(\theta)} \int w(\theta)R(\theta)\mathrm{d}\theta, \tag{4.25}$$

where B and C are free constants. Since ϕ_3 has to be bounded at infinity, we obtain two conditions one of which fixes $C = 0$ and the other one reads

$$k^2 = \frac{7\lambda_0}{20\gamma_0'\lambda_0 - 48\lambda_0'\gamma_0}. \tag{4.26}$$

This result can be regarded as a stability condition for the soliton solution (4.17) to the equation hierarchy (up to $O(\epsilon^2)$). The only stable soliton solution of (4.13) is the one with wave speed given by (4.26) because the others will give rise to an exponentially growing at infinity $O(\epsilon^2)$ correction to the solution. The solution for ϕ_3 is then given by

$$\begin{aligned}\phi_3(\theta) = &-\frac{A}{4} + \left(B - \frac{3A}{4}\right)\operatorname{sech}^2\theta\tanh\theta + \frac{3A}{4}\operatorname{sech}^2\theta \\ &+ \frac{144k^3}{7\lambda_0}\left(\frac{\lambda_0'\gamma_0}{\lambda_0} - \gamma_0'\right)\tanh\theta\operatorname{sech}^2\theta\ln(\cosh x).\end{aligned} \tag{4.27}$$

A can be fixed to 0 from the condition that the dispersion is undisturbed ahead of the soliton: $\phi_3 \to 0$ as $\theta \to \infty$. The only free constant that still remains is B. It will be determined from the solvability condition for the equation defining the next approximation ϕ_4. Unfortunately, for some important cases, like air bubbles in water, the condition (4.26) cannot be satisfied by the coefficients $\lambda_0, \lambda_0', \gamma_0$ and γ_0' in the sense that its right-hand side is negative. This means that in this case the weakly nonlinear hierarchy that we derived is unstable. Hayakawa (1994) derived a similar condition in the case of fluidized beds. In contrast to the present situation it can be satisfied in most typical physical situations and thus the soliton solution for such systems can be stable. The reason for this difference seems to be that the dispersive effect in bubbly flows can be much less strong (of course for some physical situations the solitary waves can still be stable). Lammers & Biesheuvel (1996) also indicate that their 'experiments show beyond doubt that the dispersive effects are negligibly small in the propagation of long-wavelength concentration waves' (they investigated the behaviour of air bubbles in water). In order to see what happens with the equation hierarchy in such situations it is useful to non-dimensionalize the momentum equation and compare the magnitude of the different terms involved. A natural choice for a characteristic length is the wavelength l of the void-fraction waves observed in the experiments. Matuszkiewicz *et al.* (1987) indicate that the typical wavelength for bubble flow is 0.3 m. The most obvious choice for a characteristic velocity is the velocity v_0 of the undisturbed uniform dispersion given by (3.1). For model (i) the dimensionless momentum equation (obtained from 2.18) then reads†

$$\begin{aligned}&\frac{\partial}{\partial t}\{\phi[\alpha v + \mu(v - U_m)]\} + \frac{\partial}{\partial x}\{\phi[\alpha v + \mu(v - U_m)]v\} - \phi\frac{\partial U_m}{\partial t} \\ &= -\frac{1}{Fr}\frac{(1-\phi_0)^2}{(1-\phi)^2}\phi(v - U_m) - \left\{\frac{\partial}{\partial\phi}\left[\phi(\alpha + \mu(\phi))H(\phi)\frac{(1-\phi)^4}{(1-\phi_0)^4}\right]\right. \\ &\quad \left. + \frac{1}{Fr}\frac{r_d}{l}H^{1/2}(\phi)\right\}\frac{\partial\phi}{\partial x} + \frac{1}{Fr}\phi + \frac{r_d}{l}\frac{\partial}{\partial x}\left[\phi(\alpha + \mu(\phi))H^{1/2}(\phi)\frac{(1-\phi)^2}{(1-\phi_0)^2}\frac{\partial v}{\partial x}\right],\end{aligned} \tag{4.28}$$

† For simplicity we assume that t, x, v and U_m are now dimensionless.

where

$$Fr = \frac{\rho_1}{\rho_1 - \rho_2} \frac{v_0^2}{gl}$$

is the Froude number and $\alpha = \rho_2/\rho_1$. For model (ii) a similar momentum equation can be derived. For the case of air bubbles of radius 0.4 mm in water the Froude number is approximately 1.8×10^{-3}, $r_d/l = 1.33 \times 10^{-3}$ and $\rho_2/\rho_1 = 1.2 \times 10^{-3}$. Thus, they all can be considered to be of the order of the small parameter $\delta = \epsilon^2$ which is equal to 1.08×10^{-3} for model (i) and 6.1×10^{-3} for model (ii). If we perform the reduction perturbation analysis taking into account the scaling of the coefficients given above we shall obtain the following equations for ϕ_2, ϕ_3 and ϕ_4:

$$\frac{\partial \phi_2}{\partial \tau} + 2\bar{\lambda}_0 \phi_2 \frac{\partial \phi_2}{\partial \eta} = 0, \tag{4.29}$$

$$\frac{\partial \phi_3}{\partial \tau} + 2\bar{\lambda}_0 \frac{\partial(\phi_2\phi_3)}{\partial \eta} = -\delta \frac{\partial^2 \phi_2}{\partial \eta^2}, \tag{4.30}$$

$$\frac{\partial \phi_4}{\partial \tau} + 2\bar{\lambda}_0 \frac{\partial(\phi_2\phi_4)}{\partial \eta} + \bar{\lambda}_0 \frac{\partial \phi_3^2}{\partial \eta} + \bar{\lambda}_0'' \frac{\partial \phi_2^3}{\partial \eta} = -\delta \frac{\partial^2 \phi_3}{\partial \eta^2}. \tag{4.31}$$

Here $\bar{\lambda}_0$ corresponds to the dimensionless equation (4.28) and $\bar{\lambda}_0''$ is the dimensionless equivalent to the coefficient λ_0'' (the same for both models i and ii) which is given in Appendix B. The equation for $\psi_1 = \phi_2 + \epsilon\phi_3$ is given by

$$\frac{\partial \psi_1}{\partial \tau} + 2\lambda_0 \psi_1 \frac{\partial \psi_1}{\partial \eta} = O(\epsilon^2) \tag{4.32}$$

and the equation for $\psi_2 = \phi_2 + \epsilon\phi_3 + \epsilon^2\phi_4$ by

$$\frac{\partial \psi_2}{\partial \tau} + (2\lambda_0\psi_2 + 3\lambda_0''\psi_2^2\epsilon^2) \frac{\partial \psi_2}{\partial \eta} = O(\epsilon^3). \tag{4.33}$$

The last two equations are particular cases of the equation

$$\frac{\partial \psi}{\partial \tau} + c(\psi) \frac{\partial \psi}{\partial \eta} = 0 \tag{4.34}$$

which is the kinematic-wave equation (see e.g. Whitham 1974). Its solution is given by

$$\psi = f(X), \quad \eta = X + F(X)\tau, \quad F(X) = c(f(X)), \tag{4.35}$$

where $f(\eta)$ is the distribution of ψ at $\tau = 0$. The same equation, under the assumption $Fr \ll 1$ and $Re \ll 1$ is derived by Lammers & Biesheuvel (1996). In order to verify the applicability of (4.34) to the initial stage of the development of periodic waves in the mixture we compared the numerical solution of the original equations (2.21)–(2.23), with a sinusoidal initial disturbance of the void fraction, to the solution of (4.32) (see figure 4). Both solutions practically coincide until $t = 80$ when they approach a discontinuous function and the numerical algorithm, based on Fourier spectral method, fails to integrate the equations adequately anymore. However, it is clear (from the general theory of equations of type (4.34)) that in a finite time the solution will become discontinuous. When its slope becomes large one should consider the next equations of the hierarchy since they may contain terms that can be comparable in magnitude with the terms in (4.32). Indeed, the equation for

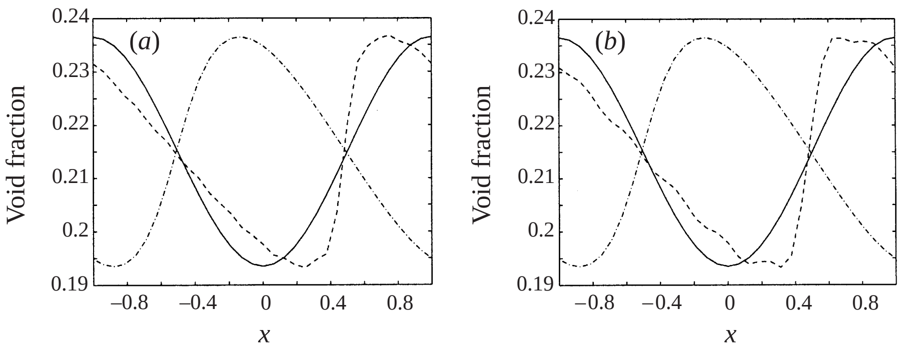

FIGURE 4. Solution for the void fraction using (*a*) the original system (2.21)–(2.23), model (ii) and (*b*) the equation (4.32): the initial condition (——), solution at $t = 40$ (—·—·—) and $t = 80$ (- - - -).

$\psi_3 = \phi_2 + \epsilon\phi_3 + \epsilon^2\phi_4 + \epsilon^3\phi_5$ becomes

$$\frac{\partial \psi_3}{\partial \tau} + c(\psi_3)\frac{\partial \psi_3}{\partial \eta} = -\epsilon^3\left(\frac{\partial^2 \psi_3}{\partial \eta^2} + \lambda_0'\frac{\partial^2 \psi_3^2}{\partial \eta^2}\right) + O(\epsilon^4). \tag{4.36}$$

The corresponding equation for ϕ_5 is

$$\frac{\partial \phi_5}{\partial \tau} + \frac{\partial (c(\psi_2)\phi_5)}{\partial \eta} = -\left(\frac{\partial^2 \psi_2}{\partial \eta^2} + \lambda_0'\frac{\partial^2 \psi_2^2}{\partial \eta^2}\right) + O(\epsilon^4),$$

where $c(\psi_2) = 2\lambda_0\psi_2 + 3\lambda_0''\epsilon^2\psi_2^2$. Using the solution for ψ_2 it can be re-written in terms of $T = \tau$ and $X = \eta - c(\psi_2)\tau$ as

$$\frac{\partial \phi_5}{\partial T} = -\frac{1}{[1+F'(X)T]^2}\frac{\partial^2[f(X)+\lambda_0' f^2(X)]}{\partial X^2} + \frac{F''(X)T}{[1+F'(X)T]^3}\frac{\partial[f(X)+\lambda_0' f^2(X)]}{\partial X}.$$

The solution to this equation is given by

$$\phi_5 = \frac{\partial^2[f(X)+\lambda_0' f^2(X)]}{\partial X^2}\frac{1}{F'(X)[1+F'(X)T]}$$
$$-\frac{F''(X)}{2F'(X)}\frac{\partial[f(X)+\lambda_0' f^2(X)]}{\partial X}\left[\frac{T}{[1+F'(X)T]^2} + \frac{1}{F'(X)[1+F'(X)T]}\right].$$

It is clear that this solution contains a finite-time singularity, which is not a surprise taking into account the negative diffusion term on the right-hand side of (4.36). Note that this result is not in contradiction with the findings of Lammers & Biesheuvel (1996) (and therefore of Batchelor 1988). From the dimensionless form of the momentum equation (4.28) it is clear that the bulk modulus of elasticity is $O(1)$ rather than $O(1/Fr)$ (r_d/l and Fr are of the same order). Therefore the diffusivity $\mathscr{D}$ (as defined by equation (2.7) of Lammers and Biesheuvel 1996) will not have a contribution by the bulk elasticity and therefore will be always negative provided that the added-mass coefficient is given by (2.11). This suggests that the model should eventually be corrected and one way to do so is to use the experimental data for $\mathscr{D}$ provided by Lammers & Biesheuvel (1996) and adjust the bulk elasticity correspondingly (using the expression for μ given by the same authors). Then the weakly nonlinear waves will be described by a Burgers equation (as shown by Lammers & Biesheuvel, 1996). Sasa & Hayakawa (1992), considering a system describing the dynamics of fluidized beds (and similar to the one considered here), suggested that in the case of negligible diffusion ($Re \ll 1$) the long-term weakly nonlinear behaviour of the system is described by the Kuramoto–Sivashinsky (KS) equation

$$\frac{\partial \psi}{\partial \tau} + \lambda \frac{\partial \psi^2}{\partial \eta} = -\frac{\partial^2 \psi}{\partial \eta^2} + C_4 \frac{\partial^4 \psi}{\partial \eta^4}.$$

An examination of the expression corresponding to C_4 in the present case reveals that it will be $O(\epsilon^2)$. The reason is that the coefficient of the third-order derivative in the original equation (Sasa & Hayakawa 1992 denote it by k) is equal to zero since no dispersion terms appear in our initial system of equations. Moreover, the rest of the terms are proportional to the Froude number which in the present case is $O(\epsilon^2)$ (their parameter ξ). Thus the KS equation will degenerate to an equation similar to (4.36) and our conclusions are consistent with the theory of Sasa & Hayakawa (1992).

In an attempt to understand the long-term behaviour of the system (2.22), (4.28) we rescale the variables as

$$\bar{\tau} = t, \qquad \bar{\eta} = \epsilon^{-2}(x - \bar{c}t), \tag{4.37}$$

where $\bar{c}$ is an as yet unknown speed. Then, neglecting the terms of $O(\epsilon^2)$, we obtain from the continuity equation

$$-\bar{c}\frac{\partial \phi}{\partial \bar{\eta}} + \frac{\partial(\phi v)}{\partial \bar{\eta}} = 0 \tag{4.38}$$

and from (4.28)

$$-\bar{c}\frac{\partial[f_1(\phi)v]}{\partial \bar{\eta}} + \frac{\partial[f_1(\phi)v^2]}{\partial \bar{\eta}} = -f_2(\phi)v - f_3(\phi)\frac{\partial \phi}{\partial \bar{\eta}} + \phi + \frac{\partial}{\partial \bar{\eta}}\left[f_4(\phi)\frac{\partial v}{\partial \bar{\eta}}\right], \tag{4.39}$$

where for simplicity we used the notation

$$f_1(\phi) = \mu\phi, \quad f_2(\phi) = \frac{(1-\phi_0)^2}{(1-\phi)^2}\phi,$$

$$f_3(\phi) = \frac{\partial}{\partial \phi}\left[\phi(\alpha + \mu(\phi))H(\phi)\frac{(1-\phi)^4}{(1-\phi_0)^4}\right] + \frac{1}{Fr}\frac{r_d}{l}H^{1/2}(\phi),$$

$$f_4(\phi) = \phi(\alpha + \mu(\phi))H^{1/2}(\phi)\frac{(1-\phi)^2}{(1-\phi_o)^2}.$$

The mixture velocity U_m can be eliminated because it is divergence free and therefore constant (we do not consider imposed unsteadiness). The scaling (4.37) is chosen because it allows the diffusive term to be included in the reduced nonlinear equation (4.39). Note that these equations should eventually be satisfied by any permanent-wave solution of (2.22), (4.28) and they are *fully nonlinear* equations. In the subsequent analysis we try to establish some conditions for existence of such fully nonlinear waves of a permanent shape. From (4.38), after integration, we obtain that

$$v = \bar{c}\frac{\phi - \phi_0}{\phi}.$$

Then (4.39) yields

$$\frac{\partial g_1(\phi)}{\partial \bar{\eta}} + g_2(\phi) + f_3(\phi)\frac{\partial \phi}{\partial \bar{\eta}} - \frac{\partial}{\partial \bar{\eta}}\left[g_4(\phi)\frac{\partial \phi}{\partial \bar{\eta}}\right] = 0, \tag{4.40}$$

where

$$g_1(\phi) = -\bar{c}^2 f_1(\phi)\frac{(\phi-\phi_0)\phi_0}{\phi^2}, \quad g_2(\phi) = \bar{c} f_2(\phi)\frac{\phi-\phi_0}{\phi} - \phi, \quad g_4(\phi) = f_4(\phi)\frac{\bar{c}\phi_0}{\phi^2}.$$

Denoting $\partial\phi/\partial\bar{\eta}$ by ζ we end up with the following system of ordinary differential equations:

$$\frac{\partial\phi}{\partial\bar{\eta}} = \zeta, \tag{4.41}$$

$$\frac{\partial\zeta}{\partial\bar{\eta}} = G_1(\phi)\zeta - G_4(\phi)\zeta^2 + G_2(\phi), \tag{4.42}$$

with

$$G_1(\phi) = \frac{\partial[g_1(\phi)]/\partial\phi + f_3(\phi)}{g_4(\phi)}, \quad G_2(\phi) = \frac{g_2(\phi)}{g_4(\phi)}, \quad G_4(\phi) = \frac{\partial[g_4(\phi)]}{\partial\phi}/g_4(\phi).$$

An equation similar to (4.40) has been derived by Harris & Crighton (1994) (their equation (5.9)). As we shall see below, the most important difference between those two equations is that in the case of Harris & Crighton (1994) $G_1(\phi) = 0$. This is due to the fact that the diffusion term in their momentum equation scales as $O(1)$ and this precludes the appearance of the first-order derivative in the fully nonlinear equation for ϕ (our equation (4.40)). Note that if this term were missing then the phase path equation corresponding to (4.41)–(4.42) would read

$$\frac{1}{2}\frac{\partial\zeta^2}{\partial\phi} = G_2(\phi) - G_4(\phi)\zeta^2 \tag{4.43}$$

and it would be integrable. In the present case, generally, $G_1(\phi) \neq 0$ and therefore the integration of the path equation is not easy. However, then we can prove that the system (4.41)–(4.42) does not admit a soliton solution. If such a solution were to exist then it clearly would be an even function and its derivative an odd function. From the phase path equation (4.42) we can conclude that this would be possible only if $G_1(\phi) = 0$. Thus the soliton solution in the present case is impossible. Note that should the diffusion term in the momentum equation (4.28) scale as $O(\epsilon)$ rather than $O(\epsilon^2)$ (as it is in the case that we consider) then G_1 would be zero and the soliton solution could exist.

5. Conclusions

In the present paper we study the relations between existing multiphase flow models relevant to bubbly flows. Similarly to Prosperetti & Jones (1987) we derive a general formulation which includes most of the available models. The momentum equation of that formulation is a balance between the convection forces acting on the particles and the liquid, the drag force, the elastic resistance to compression, gravity and an effective particle dissipation. Since the formulation of the drag parameter, the bulk modulus of elasticity and the effective particle diffusivity is still on a very heuristic basis, in the ensuing analysis we primarily used expressions for bubbly flows due to Biesheuvel & Gorissen (1990). However, in the linear analysis we also studied the influence of some other formulations of the drag force and the dispersed-phase pressure. The linear analysis of the various possible models showed that only the ones formulated by Biesheuvel & Gorissen (1990) and Stuhmiller (1977) gave reasonable neutral stability curves and critical values for the void fraction.

In the subsequent weakly nonlinear analysis we derived, using the reductive perturbation method described by Hayakawa *et al.* (1994), a weakly nonlinear equation hierarchy and solved the equations for the first two approximations of the void fraction. The solvability condition for the second (non-homogeneous) equation defined the wave speed and the amplitude of the possible soliton solution of the first (Korteweg–de Vries) equation of the hierarchy. In many cases of bubbly flows, however, this condition is not satisfied and thus the solution to the nonlinear equation derived from both models (i) and (ii) is unstable for sufficiently long times. Taking into account the scaling of the terms into the dimensionless form of the momentum equation (4.28) we showed that the weakly nonlinear equation in this case degenerates into an equation similar to a Burgers equation with a negative diffusivity, which explains this instability. Furthermore it is clear that if there is experimental evidence for the existence of stable void fraction waves in the flow then the expression for the bulk modulus of elasticity should be modified. A promising step for a realistic modelling of this term is made by Lammers & Biesheuvel (1996) who measure the diffusivity of the periodic waves propagating through a bubbly flow in water. These data can eventually be used in order to correct the expression for the bulk modulus of elasticity and the resulting weakly nonlinear equation is a (stable) Burgers equation. We also established that a necessary condition for the existence of a soliton solution of the models considered here is that the diffusion term should scale at least as $O(\sqrt{Fr})$. Thus, for the most physical situations the appearance of solitons in bubbly flows is unlikely.

A question that still remains open is whether the one-dimensional weakly nonlinear analysis performed here and commonly used in the literature can give an adequate description of the development of instability in uniform multiphase flow. It is possible that the one-dimensional linear waves break immediately into two-dimensional structures. The answer to that question can be sought through extensive two-dimensional numerical simulations, which will be a subject of our further work.

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Appendix A

λ_0' and γ_0' for model (i) (Biesheuvel & Gorissen 1990) are given by

$$\begin{aligned}\lambda_0' = -\frac{1}{\beta_0}\{&\mu_{2,0}''\rho_1\phi_0^2(1-\phi_0)^2 v_0(v_0-c_0)\\ &+\mu_{2,0}'\rho_1\phi_0[2c_0v_0(1-\phi_0)-2v_0(\lambda_0\phi_0+v_0)(1-\phi_0)-2c_0(c_0-v_0)]\\ &+\mu_{2,0}\rho_1[4\lambda\phi_0(v_0-c_0)+2v_0\lambda_0\phi_0^2+2v_0(1-\phi_0)(v_0-c_0)\\ &+2c_0(v_0-c_0)\phi_0/(1-\phi_0)-2c_0(v_0-c_0)]+\xi_0'\phi_0(1-\phi_0)\\ &+\rho_2[2(v_0-c_0)^2(1-\phi_0)+4\lambda_0(v_0-c_0)\phi_0(1-\phi_0)]\},\end{aligned}$$

$$\gamma_0' = \frac{\gamma_0}{\beta_0}[2\rho_2(v_0-c_0)\phi_0(1-\phi_0)-\rho_1v_0\mu_{2,0}'\phi_0^2(1-\phi_0)-2\rho_1\mu_{2,0}v_0\phi_0(1-\phi_0)-2\rho_1\mu_{2,0}c_0\phi_0],$$

with $\mu_2 = \mu(\phi)(1-\phi)$. For model (ii) (Stuhmiller 1977) they read

$$\lambda_0' = \frac{\lambda_0}{\beta_0}[\rho_1(\mu_{1,0}+\phi_0)c_0\phi_0 + (\rho_2\phi_0+\rho_1\mu_{1,0})(c_0-v_0)(1-\phi_0)]$$
$$+\frac{1}{2\beta_0}\left[\rho_1\mu_{1,0}\frac{v_0-c_0}{\phi_0}\frac{c_0}{1-\phi_0}+\rho_1\phi_0\frac{c_0^2}{(1-\phi_0)^2}\right.$$
$$+(\rho_1\mu_{1,0}'+\rho_1)c_0^2\phi_0 - \frac{(\rho_2\phi_0+\rho_1\mu_{1,0})(c_0-v_0)^2(1-\phi_0)}{\phi_0}$$
$$\left.+(\rho_2+\rho_1\mu_{1,0}')(v_0-c_0)^2(1-\phi_0)-\xi_0'\phi_0(1-\phi_0)\right],$$
$$\gamma_0' = \frac{\gamma_0}{\beta_0}[\rho_1(\mu_{1,0}+\phi_0)c_0\phi_0 + (\rho_2\phi_0+\rho_1\mu_{1,0})(c_0-v_0)(1-\phi_0)],$$

with $\mu_1 = \mu(\phi)\phi/(1-\phi)$.

Appendix B

λ_0'' for models (i) (Biesheuvel & Gorissen 1990) and (ii) (Stuhmiller 1977) is given by

$$\lambda_0'' = \frac{(v_0-c_0)(1-\phi_0)}{\phi_0^2} - \frac{c_0\phi_0}{(1-\phi_0)^2} + \frac{1-2\phi_0}{\phi_0(1-\phi_0)} - \frac{\beta_0'}{\beta_0}\left[\frac{(v_0-c_0)(1-\phi_0)}{\phi_0}+\lambda_0\right]$$
$$-\frac{\beta_0''}{2\beta_0}[(c_0-v_0)(1-\phi_0)+c_0\phi_0] - \frac{\beta_0'''}{6\beta_0}v_0\phi_0(1-\phi_0).$$

REFERENCES

Anderson, T. B. & Jackson, R. 1967 Fluid mechanical description of fluidized beds. Equations of motion. *Indust. Engng Chem. Fundam.* **6**, 527–539.

Anderson, T. B., Sundaresan, S. & Jackson, R. 1995 Instabilities and formation of bubbles in fluidized beds. *J. Fluid Mech.* **303**, 327–366.

Batchelor, G. K. 1988 A new theory of instability of a uniform fluidized bed. *J. Fluid Mech.* **193**, 75–110.

Biesheuvel, A. & Gorissen, W. C. M. 1990 Void fraction disturbances in a uniform bubbly fluid. *Intl J. Multiphase Flow* **16**, 211–231.

Biesheuvel, A. & Spoelstra, S. 1989 The added mass coefficient of dispersions of gas bubbles in liquid. *Intl J. Multiphase Flow* **15**, 911–924.

Biesheuvel, A. & Wijngaarden, L. van 1984 Two-phase flow equations for a dilute dispersion of gas bubbles in liquid. *J. Fluid Mech.* **148**, 301–318.

Drew, D. A. & Segel, L. A. 1971 Averaged equations for two-phase flows. *Stud. Appl. Maths* **50**, 205–231.

Garg, S. K. & Pritchett, J. W. 1975 Dynamics of gas-fluidized beds. *J. Appl. Phys.* **46**, 4493–4500.

Harris, S. E. & Crighton, D. G. 1994 Solitons, solitary waves, and voidage disturbances in gas-fluidized beds. *J. Fluid Mech.* **266**, 243–276.

Hayakawa, H. 1994 Pseudo-soliton in fluidized beds and its perturbed treatment. *Teoreticheskaya i Matematicheskaya Fizika* **99**, 309–317.

Hayakawa, H., Komatsu, T. S. & Tsuzuki, T. 1994 Pseudo-solitons in fluidized beds. *Physica* A **204**, 277–289.

Homsy, G. M., El-Kaissy, M. M. & Didwania, A. 1980 Instability waves and the origin of bubbles in fluidized beds. Part II. Comparison with theory. *Intl J. Multiphase Flow* **6**, 305–318.

Ishii, M. 1975 *Thermo-fluid Dynamics Theory of Two-Phase Flow*. Paris: Eyrolles.

Jackson, R. 1971 Fluid mechanical theory. In *Fluidization* (ed. J. F. Davidson & D. Harrison). Academic.

Jones, A. V. & Prosperetti, A. 1985 On the suitability of first-order differential models for two-phase flow prediction. *Intl J. Multiphase Flow* **11**, 133–148.

Lammers, J. H. & Biesheuvel, A. 1996 Concentration waves and the instability of bubbly flows. *J. Fluid Mech.* **328**, 64–94.

Liu, J. T. C. 1982 Note on a wave-hierarchy interpretation of fluidized bed instabilities. *Proc. R. Soc. Lond.* A **380**, 229–239.

Liu, J. T. C. 1983 Nonlinear unstable wave disturbances in fluidized beds. *Proc. R. Soc. Lond.* A **389**, 331–347.

Matuszkiewicz, A., Flamand, J. C. & Bouré, J. A. 1987 The bubble-slug flow pattern transition and instabilities of void fraction waves. *Intl J. Multiphase Flow* **13**, 199–217.

Needham, D. J. & Merkin, J. H. 1983 The propagation of a voidage disturbance in a uniformly fluidized bed. *J. Fluid Mech.* **131**, 427–454.

Nigmatulin, R. I. 1979 Spatial averaging in the mechanics of heterogeneous and dispersed systems. *Intl J. Multiphase Flow* **5**, 353–385.

Prosperetti, A. & Jones, A. V. 1987 The linear stability of general two-phase flow models-II. *Intl J. Multiphase Flow* **13**, 161–171.

Richardson, J. F. 1971 Incipient fluidization and particulate systems. In *Fluidization* (ed. J. F. Davidson & D. Harrison). Academic.

Sangani A. S. & Didwania, A. K. 1993 Dispersed-phase stress tensor in flows of bubbly liquids. *J. Fluid Mech.* **248**, 27–54.

Sasa Shin-ichi & Hayakawa, H. 1992 Void-fraction dynamics in fluidization. *Europhys. Lett.* **17**, 685–689.

Stuhmiller, J. H. 1977 The influence of interfacial pressure forces on the character of two-phase flow model equations. *Intl J. Multiphase Flow* **3**, 551–560.

Whitham, G. B. 1974 *Linear and Nonlinear Waves*. John Wiley & Sons.

Wijngaarden, L. van & Kapteyn, C. 1990 Concentration waves in dilute bubble/liquid mixtures. *J. Fluid Mech.* **212**, 111–137.

Zhang D. Z. & Prosperetti, A. 1994 Averaged equations for inviscid dispersed two-phase flow. *J. Fluid Mech.* **267**, 185–219.

Zuber, N. 1964 On the dispersed two-phase flow in the laminar flow regime. *Chem. Engng Sci.* **19**, 897–917.

Zuber, N. & Ishii, N. 1979 Drag coefficient and relative velocity in bubbly, droplet or particulate flows. *AIChE J.* **25**, 843–855.